# Exercise 69

Find a cubic function $y = ax^3 + bx^2 + cx + d$ whose graph has horizontal tangents at the points $(-2, 6)$ and $(2, 0)$.

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**Solution**

Take the derivative of the cubic function.

$$\begin{aligned} y' &= \frac{d}{dx}(ax^3 + bx^2 + cx + d) \\ &= \frac{d}{dx}(ax^3) + \frac{d}{dx}(bx^2) + \frac{d}{dx}(cx) + \frac{d}{dx}(d) \\ &= a\frac{d}{dx}(x^3) + b\frac{d}{dx}(x^2) + c\frac{d}{dx}(x) + \frac{d}{dx}(d) \\ &= a(3x^2) + b(2x) + c(1) + (0) \\ &= 3ax^2 + 2bx + c \end{aligned}$$

The graph of $y$ has horizontal tangents at $x = -2$ and $x = 2$, so the derivative must be equal to zero for these values of $x$.

$$3a(-2)^2 + 2b(-2) + c = 0$$

$$3a(2)^2 + 2b(2) + c = 0$$

Simplify the system

$$12a - 4b + c = 0$$

$$12a + 4b + c = 0$$

and solve it for $a$ and $b$ in terms of $c$.

$$a = -\frac{c}{12} \qquad b = 0$$

Consequently, the cubic function with horizontal tangents at $x = -2$ and $x = 2$ is

$$y = -\frac{c}{12}x^3 + cx + d.$$

Use the fact that $y = 6$ when $x = -2$ and the fact that $y = 0$ when $x = 2$ to determine $c$ and $d$.

$$y(-2) = -\frac{c}{12}(-2)^3 + c(-2) + d = 6$$

$$y(2) = -\frac{c}{12}(2)^3 + c(2) + d = 0$$

Solve this system of equations for $c$ and $d$.

$$c = -\frac{9}{4} \qquad d = 3$$

Therefore, the cubic function whose graph has horizontal tangents at the points $(-2, 6)$ and $(2, 0)$ is

$$y = \frac{3}{16}x^3 - \frac{9}{4}x + 3.$$